WNE Linear Algebra
Final Exam
Series B

6 February 2021

Questions

Please use a single file for all questions. Give reasons to your answers or provide a counterexample. Please provide the following data in the pdf file

- **name, surname and your student number,**
- **number of your group,**
- **number of the corresponding problem and the series.**

Each question is worth 4 marks.

Question 1.
Let $q\colon \mathbb{R}^2 \to \mathbb{R}$ be a quadratic form given by a symmetric matrix $M \in M(2 \times 2; \mathbb{R})$, i.e., $M = M^\intercal$ and

$$q((x,y)) = \begin{bmatrix} x & y \end{bmatrix} M \begin{bmatrix} x \\ y \end{bmatrix}.$$

If $\det M = 0$ does there exist a vector $(x,y) \in \mathbb{R}^2,\ (x,y) \neq \mathbf{0}$ such that

$$q((x,y)) = 0?$$

Answer 1.
Yes, it does exist. If $\det M = 0$ the columns of matrix M are linearly independent, i.e., there exists a non–zero(!) vector $v \in \mathbb{R}^2$ such that

$$Mv = \mathbf{0}.$$

In particular,

$$v^\intercal M v = v^\intercal (Mv) = 0.$$

If you are not convinced, check Kronecker–Capelli Theorem (Lecture 7).

In down–to–earth terms, if $M = \begin{bmatrix} a & b \\ b & c \end{bmatrix}$ and $v = \begin{bmatrix} c \\ -b \end{bmatrix}$, then

$$Mv = \begin{bmatrix} \det M \\ 0 \end{bmatrix} = 0.$$

In particular $q(v) = 0$. If $c = b = 0$, take $v = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

Question 2.
Let $V \subset \mathbb{R}^3$ be a proper subspace of $\mathbb{R}^3$ (i.e., $\dim V = 1$ or $\dim V = 2$). Let

$$P_V : \mathbb{R}^3 \to \mathbb{R}^3,$$

denote the (linear) orthogonal projection onto V and let

$$S_V : \mathbb{R}^3 \to \mathbb{R}^3,$$

denote the (linear) orthogonal reflection about V. Does it follow that

$$S_V \circ P_V = P_V \circ S_V?$$

Answer 2.
Yes, it does. For any $v \in \mathbb{R}^3$ let

$$v = w + u,$$

where $w \in V$ and $u \in V^\perp$. Then

$$P_V(v) = P_V(v)(w + u) = w,$$
$$P_V(w - u) = w,$$
$$S_V(w) = w,$$
$$S_V(v) = w - u,$$

hence

$$S_V(P_V(v)) = S_V(w) = w,$$
$$P_V(S_V(v)) = P_V(w - u) = w.$$

Question 3.
If $A, B \in M(2 \times 2; \mathbb{R})$, $\det A > 0$, $\det B > 0$ does it follow that

$$\det(A^2) + 2\det(AB) + \det(B^2) > 0?$$

Answer 3.
Yes, it does.

$$\det(A^2) + 2\det(AB) + \det(B^2) = (\det A + \det B)^2 > 0.$$

Question 4.
Let $A, B \in M(m \times n; \mathbb{R})$ be matrices such that

$$A \xrightarrow{c_i + \alpha c_j} B,$$

for some $i, j \in \{1, 2, \ldots, n\}$ and some $\alpha \in \mathbb{R}$ (i.e., one is obtained from the other by the column operation $c_i + \alpha c_j$). Does it follow that the reduced (row) echelon form of matrix A is the same as the reduced (row) echelon form of matrix B?

Answer 4.
No, it does not. Consider

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \xrightarrow{c_3 + c_1} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix},$$

where matrices A and B are in the reduced row echelon form.

Question 5.
Let $A, B \in M(2 \times 2; \mathbb{R})$ be two matrices and let vector $v \in \mathbb{R}^2$ be an eigenvector of matrix A and simultaneously an eigenvector of matrix B (corresponding to possibly different eigenvalues). Is vector $Bv \in \mathbb{R}^2$ an eigenvector of matrix A?

Answer 5.
No, it isn't. If $Bv = \mathbf{0}$ then Bv is not an eigenvector (since it is the zero vector).